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A new sufficient condition for a digraph to be Hamiltonian

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Abstract

In Bang-Jensen et al. (Sufficient conditions for a digraph to be Hamiltonian, J. Graph Theory 22 (1996) 181–187) the following extension of Meyniels theorem was conjectured: If D is a strongly connected digraph on n vertices with the property that $d(x)+d(y)\geqslant 2n-1$ for every pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian. We verify the conjecture in the special case where we also require that $\min\{d^+(x)+d^-(y),\ d^-(x)+d^+(y)\}\geqslant n-1$ for all pairs of vertices x, y as above. This generalizes one of the results in [2]. Furthermore we provide additional support for the conjecture above by showing that such a digraph always has a factor (a spanning collection of disjoint cycles). Finally, we show that if D satisfies that $d(x)+d(y)\geqslant \frac{5}{2}n-4$ for every pair of non-adjacent vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.

Keywords: Hamiltonian cycle; Hamiltonian digraph; Sufficient condition; Hamiltonian path; Meyniels theorem; Locally semicomplete digraph; In-semicomplete digraph; Degree condition

1. Introduction

For convenience of the reader we provide all necessary terminology and notation in one section, Section 2.

While there are many degree conditions which guarantee that an undirected graph is Hamiltonian, not so many conditions are known to be sufficient to guarantee that a digraph is Hamiltonian. In each of the conditions below D is a strongly connected digraph on n vertices:

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Theorem 1.1 (Ghouila-Houri [6]). *If $d(x) \geqslant n$ for all vertices $x \in V(D)$, then D is Hamiltonian.*

Theorem 1.2 (Woodall [10]). *If $d^+(x) + d^-(y) \geqslant n$ for all pairs of vertices x and y such that there is no arc from x to y, then D is Hamiltonian.*

Theorem 1.3 (Meyniel [9]). *If $d(x) + d(y) \geqslant 2n - 1$ for all pairs of non-adjacent vertices in D then D is Hamiltonian.*

It is easy to see that Theorem 1.3 implies Theorems 1.1 and 1.2. For a short proof of Theorem 1.3 see [3,5].

Theorem 1.4 (Manoussakis [8]). *Suppose that D satisfies the following condition for every triple $x, y, z \in V(D)$ such that x and y are non-adjacent: If there is no arc from x to z, then $d(x) + d(y) + d^+(x) + d^-(z) \geqslant 3n - 2$. If there is no arc from z to x then $d(x) + d(y) + d^-(x) + d^+(z) \geqslant 3n - 2$. Then D is Hamiltonian.*

Each of these theorems imposes a degree condition on all pairs of non-adjacent vertices. In [2,12] it was shown that it is possible to weaken the rather strong demand of high degree for every pair of non-adjacent vertices, by requiring this only for some pairs of non-adjacent vertices.

Theorem 1.5 (Bang-Jensen [2]). *Suppose that $\min\{d(x), d(y)\} \geqslant n - 1$ and $d(x) + d(y) \geqslant 2n - 1$ for any pair of non-adjacent vertices x, y with a common in-neighbour, then D is Hamiltonian.*

Theorem 1.6 (Bang-Jensen [2]). *Suppose that $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geqslant n$ for every pair of vertices x, y with a common out-neighbour or a common in-neighbour, then D is Hamiltonian.*

Note that Theorems 1.5 and 1.6 generalize Theorems 1.1 and 1.2, respectively.

Theorem 1.7 (Zhao and Meng [12]). *Suppose that $d^+(x) + d^+(y) + d^-(u) + d^-(v) \geqslant 2n - 1$ for every choice of vertices x, y, u, v (not necessarily all distinct) such that x, y are non-adjacent and have a common out-neighbour and u, v are non-adjacent and have a common in-neighbour, then D is Hamiltonian.*

None of Theorems 1.3, 1.5–1.7 imply each other. The conditions in Theorems 1.5 and1.6 where inspired by the fact that if D has no pair of non-adjacent vertices with a common in-neighbour or a common out-neighbour, then D is a locally semicomplete digraph and it was shown in [1] that every strongly connected locally semicomplete digraph is Hamiltonian.

In [2] the following generalization of Theorem 1.3 was proposed

Conjecture 1.8 (Bang-Jensen [2]). *Let D be a strong digraph. Suppose that $d(x) + d(y) \geqslant 2n - 1$ for every pair of non-adjacent vertices $\{x, y\}$ with a common out-neighbour or a common in-neighbour. Then D is Hamiltonian.*

This conjecture seems quite difficult to prove, but we are able to prove it in the special case when we also require that $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geqslant n - 1$ (Theorem 3.1). Note that already this generalizes Theorem 1.6 and as we show in Section 4 there are digraphs which satisfy this new condition and none of the previous conditions which where proved sufficient for hamiltonicity. In Section 5 we provide additional support for Conjecture 1.8 by showing that every digraph satisfying the condition of Conjecture 1.8 has a factor. We also show that if we replace the degree condition $d(x)+d(y) \geqslant 2n-1$ by $d(x)+d(y) \geqslant \frac{5}{2}n-4$ in Conjecture 1.8, then D is Hamiltonian.

2. Terminology and notation

We shall assume that the reader is familiar with the standard terminology on digraphs and refer to [4] for terminology not discussed here.

If D is a digraph and $X \subset V(D)$, then we denote the subdigraph of D *induced* by X by $D\langle X \rangle$.

Every *cycle* and *path* is assumed to be simple and directed. The *length* of a cycle or path is the number of its arcs. D always denotes a digraph with n vertices and vertex set $V(D)$. The digraph D is *Hamiltonian* if it contains a *Hamiltonian cycle*, namely a cycle of length n. A *factor* of D is a spanning collection of vertex disjoint cycles in D.

Let x, y be distinct vertices in D. If there is an arc from x to y then we say that x *dominates* y and write $x \to y$ and call y (respectively, x) an *out-neighbour* (respectively, an *in-neighbour*) of x (respectively, y). We let $N^+(x), N^-(x)$ denote the set of out-neighbours, respectively the set of in-neighbours of x in D. Similarly, define $N(x)$ to be $N(x) = N^+(x) \cup N^-(x)$ and extend that to subsets of $V(D)$ by $N(X) = (\bigcup_{\{x \in X\}} N(x)) \setminus X$.

If A and B are disjoint subsets of $V(D)$ and there are no arcs from B to A, then we denote this by $A \Rightarrow B$. D is an *out-semicomplete digraph* (*in-semicomplete digraph*) if D has no pair of non-adjacent vertices with a common in-neighbour or a common out-neighbour. D is a *locally semicomplete digraph* if D is both out-semicomplete and in-semicomplete.

If $x \in V(D)$ and H is a subgraph of D, the *in-degree* $d_H^-(x)$ (*out-degree* $d_H^+(x)$) of x with respect to H is the number of vertices in H dominating x (dominated by x, respectively). The *degree* of x with respect to H is $d_H(x) = d_H^-(x) + d_H^+(x)$. When $H = D$, the subscript H will be omitted.

If x and y are distinct vertices of D and P is a path from x to y, we say that P is an (x, y)-*path*. If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y, $C[x,y]$ denotes the subpath of C from x to y.

Let C be a cycle in D. An (x,y)-path P is a *C-bypass* if $|V(P)| \geqslant 3$, $x \neq y$ and $V(P) \cap V(C) = \{x,y\}$. The *gap of P with respect to C* is the length of the path $C[x,y]$.

D is *strongly connected* (or just *strong*) if there exists an (x,y)-path in D for every ordered pair of distinct vertices $\{x,y\}$ of D.

Let $P = u_1u_2\dots u_s$ be a path in D (possibly, $s=1$) and let $Q = v_1v_2\dots v_t$ be a path in $D - V(P)$. P has a *partner* on Q if there is an arc (the *partner of P*) $v_i \to v_{i+1}$ on Q such that $v_i \to u_1$ and $u_s \to v_{i+1}$. In this case the path P can be inserted into Q to give a new (v_1,v_t)-path $Q[v_1,v_i]PQ[v_{i+1},v_t]$. P has a *collection of partners* on Q if there are integers $i_1 = 1 < i_2 < \cdots < i_m = s+1$ such that, for every $k = 2,3,\dots,m$, the subgraph $P[u_{i_{k-1}}, u_{i_k - 1}]$ has a partner on Q.

We conclude this section with some results from [2]

Lemma 2.1 (Bang-Jensen [2]). *Let P be a path in D and let $Q = v_1v_2\dots v_t$ be a path in $D - V(P)$. If P has a collection of partners on Q, then there is a (v_1,v_t)-path R in D so that $V(R) = V(P) \cup V(Q)$.*

Lemma 2.2 (Bang-Jensen [2] and Bondy and Thomassen [5]). *Let $Q = v_1v_2\dots v_t$ be a path in D and let w, w' be vertices of $V(D) - V(Q)$ (possibly $w = w'$). If there do not exist consecutive vertices v_i, v_{i+1} on Q such that $v_i \to w$, $w' \to v_{i+1}$ are arcs of D, then $d_Q^-(w) + d_Q^+(w') \leqslant t + \xi$, where $\xi = 1$ if $v_t \to w$ and 0, otherwise.*

In the special case when $w' = w$ above, we get the following interpretation of the statement of Lemma 2.2.

Lemma 2.3 (Bang-Jensen [2]). *Let $Q = v_1v_2\dots v_t$ be a path in D, and let $w \in V(D) - V(Q)$. If w has no partner on Q, then $d_Q(w) \leqslant t+1$. If, in addition, v_t does not dominate w, then $d_Q(w) \leqslant t$.*

3. Main result

Theorem 3.1. *Let D be a strong digraph with $n \geqslant 2$ vertices. Suppose that $d(x) + d(y) \geqslant 2n-1$ and $\min\{d^+(x) + d^-(y), d^-(x) + d^+(y)\} \geqslant n-1$ for every pair of non-adjacent vertices $x, y \in V(D)$ with a common in-neighbour or a common out-neighbour. Then D is Hamiltonian.*

Proof. Since D is strong, it contains a cycle. Let $C = x_1x_2\dots x_mx_1$ be a longest cycle in D and suppose, to the contrary, that $m < n$. Set $R = D - V(C)$. We first prove the following claim:

Claim 1. *Let y be a vertex of R. If there are two different vertices x_α and x_β on C such that $x_\alpha \to y \to x_\beta$ and y is not adjacent with any vertex in $C - V(C[x_\beta, x_\alpha])$,* *then the following holds*:

$$|V(C')| \geqslant 1, \tag{1}$$

$$d_C(y) = |V(C'')| + 1, \tag{2}$$

$$d^+_{C''}(x_{\beta-1}) + d^-_{C''}(x_{\alpha+1}) = |V(C'')| + 1, \tag{3}$$

$$d_R(y) + d^+_R(x_{\beta-1}) + d^-_R(x_{\alpha+1}) = 2(n-m-1), \tag{4}$$

$$d^+_{C'}(x_{\beta-1}) + d^-_{C'}(x_{\alpha+1}) = 2(|V(C')| - 1), \tag{5}$$

where $C' = C[x_{\alpha+1},\ x_{\beta-1}]$ and $C'' = C[x_\beta,\ x_\alpha]$.

Proof. Since C is a longest cycle in D, the vertex y cannot be inserted in C. Hence, $|V(C')| \geqslant 1$ and

$$d_C(y) \leqslant |V(C'')| + 1 \tag{6}$$

by Lemma 2.3. Because C' has no partner on C'', we conclude by Lemma 2.2 that

$$d^+_{C''}(x_{\beta-1}) + d^-_{C''}(x_{\alpha+1}) \leqslant |V(C'')| + 1. \tag{7}$$

It is also a simple matter to check that for every vertex $z \in V(R-y)$, at most two of the arcs $z \to y$, $y \to z$, $x_{\beta-1} \to z$, $z \to x_{\alpha+1}$ can be arcs of D. It follows that

$$d_R(y) + d^+_R(x_{\beta-1}) + d^-_R(x_{\alpha+1}) \leqslant 2(n-m-1). \tag{8}$$

Obviously, we also have

$$d^+_{C'}(x_{\beta-1}) + d^-_{C'}(x_{\alpha+1}) \leqslant 2(|V(C')| - 1). \tag{9}$$

If one of the four inequalities (6)–(9) is strict, then we have

$$\begin{aligned} d^+(y) + d^-(x_{\alpha+1}) + d^-(y) + d^+(x_{\beta-1}) &= d(y) + d^-(x_{\alpha+1}) + d^+(x_{\beta-1}) \\ &< (|V(C'')| + 1) + 2(n-m-1) \\ &\quad +(|V(C'')| + 1) + 2(|V(C')| - 1) \\ &= 2n - 2, \end{aligned}$$

this is a contradiction to the assumption of the theorem which implies that $d^+(y) + d^-(x_{\alpha+1}) \geqslant n-1$ and $d^-(y) + d^+(x_{\beta-1}) \geqslant n-1$. $\square$

Now, we show that D contains a C-bypass. Since D is strong, there is a vertex y in R such that $y \to x_i$ for some $i \in \{1, 2, \ldots, m\}$. If y dominates every vertex on C, then a path P from a vertex x_j on C to y such that $V(P) \cap V(C) = \{x_j\}$ together with the arc $y \to x_{j+1}$ and the path $C[x_{j+1}, x_j]$ form a longer cycle in D, a contradiction. Therefore, C has a vertex x_k such that $y \to x_k$ and y and x_{k-1} are non-adjacent with x_k as a common out-neighbour. Since $d^+(x_{k-1}) + d^-(y) \geqslant n-1$, there is a vertex z

such that $x_{k-1} \to z \to y$. It is easy to see that $z \in V(C)$. Let x_ℓ be a vertex on C such that $x_\ell \to y$. If we can choose $\ell \neq k$, then $x_\ell y x_k$ is a C-bypass. In the remaining case when the only choice for x_ℓ has $\ell = k$, we have $d^+(x_{k-1}) + d^-(y) = n-1$. Then it follows from $d(x_{k-1}) + d(y) \geqslant 2n-1$ that $d^-(x_{k-1}) + d^+(y) \geqslant n$. The last inequality implies that there is a vertex $w \in V(D - \{x_{k-1}, x_k, y\})$ such that $y \to w \to x_{k-1}$, and hence, $x_k y w$ if $w \in V(C)$ or $x_k y w x_{k-1}$ if $w \notin V(C)$ is a C-bypass.

Let $P = u_1 u_2 \dots u_s$ ($s \geqslant 3$) be a C-bypass with minimum gap among the gaps of all C-bypasses. Assume without loss of generality, that P is minimal with respect to the minimum gap and $u_1 = x_1$ and $u_s = x_\gamma$ with $2 \leqslant \gamma \leqslant m$. In fact, $\gamma \geqslant s \geqslant 3$ since C is a longest cycle in D. Note that, by our choice of P, the vertex $u_1 = x_1$ is a common in-neighbour of the non-adjacent vertices u_2, x_2. Similarly, u_{s-1}, $x_{\gamma-1}$ are non-adjacent with a common out-neighbour.

Let $C' = C[x_2, x_{\gamma-1}]$ and $C'' = C[x_\gamma, x_1]$. By the choice of the C-bypass P, we see that no vertex of $P[u_2, u_{s-1}]$ is adjacent with any vertex of C'. Moreover, for every vertex $z \in R - u_{s-1}$, the inequality $d_z(u_{s-1}) + d_z(x_{\gamma-1}) \leqslant 2$ holds, and hence, $d_R(u_{s-1}) + d_R(x_{\gamma-1}) \leqslant 2(n-m-1)$. Since u_{s-1} cannot be inserted in C, we have $d_{C''}(u_{s-1}) \leqslant |V(C'')| + 1$ by Lemma 2.3. From the last two inequalities and the assumption of the theorem, we see that

$$\begin{aligned}
2n - 1 &\leqslant d(u_{s-1}) + d(x_{\gamma-1}) \\
&= d_{C''}(u_{s-1}) + d_R(u_{s-1}) + d_R(x_{\gamma-1}) + d_{C'}(x_{\gamma-1}) + d_{C''}(x_{\gamma-1}) \\
&\leqslant (|V(C'')| + 1) + 2(n-m-1) + 2(|V(C')| - 1) + d_{C''}(x_{\gamma-1}) \\
&= 2n - |V(C'')| - 3 + d_{C''}(x_{\gamma-1}).
\end{aligned}$$

It follows that

$$d_{C''}(x_{\gamma-1}) \geqslant |V(C'')| + 2. \tag{10}$$

Similarly, we can deduce that

$$d_{C''}(x_2) \geqslant |V(C'')| + 2. \tag{11}$$

So, by Lemma 2.3, x_2 (respectively, $x_{\gamma-1}$) can be inserted in C''. If $\gamma \leqslant 4$, then, by Lemma 2.1, we can insert x_2 and $x_{\gamma-1}$ in the part C'' of the cycle $x_1 u_2 u_3 \dots x_\gamma x_{\gamma+1} \dots x_m x_1$ and get a cycle of length at least $m+1$. Therefore,

$$\gamma \geqslant 5. \tag{12}$$

Furthermore, if $m = \gamma$, then we see from (10) (respectively, (11)) $x_{\gamma-1} \to x_1$ (respectively, $x_m \to x_2$) and we get a cycle longer than C. Hence,

$$m \geqslant \gamma + 1. \tag{13}$$

Suppose that $s \geqslant 4$. We first prove the following claim.

Claim 2. *There is a vertex x_α with $\gamma + 1 \leqslant \alpha \leqslant m$ such that $x_\alpha \to u_{s-1}$ and u_{s-1} is not adjacent with any vertex of $C[x_{\alpha+1}, x_{\gamma-1}]$.*

Proof. Because of $d^+(x_{\gamma-1}) + d^-(u_{s-1}) \geqslant n-1$, there is a vertex z such that $x_{\gamma-1} \to z \to u_{s-1}$ and it is clear that $z \in V(C)$. Let x_α be a vertex such that $x_\alpha \to u_{s-1}$ and there is no arc from $C[x_{\alpha+1}, x_{\gamma-1}]$ to u_{s-1}. Clearly, $\gamma \leqslant \alpha \leqslant m$. Note that we will have $\alpha \neq \gamma$, unless x_γ is the only in-neighbour of u_{s-1} on C''.

If $\alpha = \gamma$, then $d^+(x_{\gamma-1}) + d^-(u_{s-1}) = n-1$ and we conclude from $d(x_{\gamma-1}) + d(u_{s-1}) \geqslant 2n-1$ that there is a vertex $w \neq x_\gamma$ with $u_{s-1} \to w \to x_{\gamma-1}$. Obviously, $w \in V(C)$ and w belongs to $C[x_{\gamma+2}, x_1]$.

If $\alpha \geqslant \gamma + 1$ and there is some vertex $x_{\beta'}$ on $C[x_{\alpha+1}, x_{\gamma-1}]$ which is adjacent with u_{s-1}, then, by the choice of α, u_{s-1} dominates $x_{\beta'}$ and $x_{\beta'}$ belongs to $C[x_{\alpha+2}, x_1]$.

Therefore, if the claim is not true, then D contains a C-bypass $x_\alpha \to u_{s-1} \to x_\beta$ for some x_β on $C[x_{\alpha+2}, x_1]$ such that u_{s-1} is not adjacent with any vertex on $C[x_{\alpha+1,\beta-1}]$. According to equality (2), we have $d_Q(u_{s-1}) = |V(Q)| + 1$, where $Q = C[x_\beta, x_\alpha]$. Note that $C' \subseteq Q$. Let $Q' = C[x_\beta, x_1]$ and $Q'' = C[x_\gamma, x_\alpha]$. Because of (12) and the fact that u_{s-1} is not adjacent with any vertex on C', at least one of the two inequalities $d_{Q'}(u_{s-1}) \geqslant |V(Q')| + 2$ and $d_{Q''}(u_{s-1}) \geqslant |V(Q'')| + 2$ holds. So, u_{s-1} can be inserted into Q' or into Q'' by Lemma 2.3, a contradiction. $\square$

Similarly, we can show the following claim:

Claim 3. *There is a vertex x_β with $\gamma + 1 \leqslant \beta \leqslant m$ such that $u_2 \to x_\beta$ and u_2 is not adjacent with any vertex of $C[x_2, x_{\beta-1}]$.*

Considering the C-bypass $x_\alpha \to u_{s-1} \to x_\gamma$ (respectively, $x_1 \to u_2 \to x_\beta$) and applying equality (5), we see that $x_{\gamma-1}$ dominates all vertices of $C[x_{\alpha+1}, x_1]$ (respectively, $x_{\beta-1} \to x_2$).

If $\alpha \geqslant \beta$, then $u_{s-1}C[x_{\gamma,\beta-1}]x_2C[x_3, x_{\gamma-1}]C[x_{\alpha+1,1}]u_2C[x_\beta, x_\alpha]u_{s-1}$ is a cycle of length $m+2$, a contradiction.

If $\alpha < \beta$, then $P[u_2, u_{s-1}]C[x_{\gamma,\beta-1}]x_2C[x_2, x_{\gamma-1}]C[x_{\beta,1}]u_2$ is a cycle of length $m + (s-2)$, a contradiction.

Finally, suppose that $s = 3$. By (5), we see that C' has a Hamiltonian path $x_{\gamma-1}x_3 \ldots x_{\gamma-2}x_2$. Hence, by the maximality of C and Lemma 2.2, $d^+_{C''}(x_2) + d^-_{C''}(x_{\gamma-1}) \leqslant |V(C'')| + 1$. It follows by (3) that $d_{C''}(x_2) + d_{C''}(x_{\gamma-1}) \leqslant 2(|V(C'')| + 1)$, a contradiction to (10) and (11). $\square$

4. The independence of our condition

Consider the infinite family $D(r,s)$, $r,s \geqslant 6$ of digraphs in Fig. 1. Note that all pairs of non-adjacent vertices with a common in-, or out-neighbour are of the form x and a vertex from $V(K_r^*) \cup V(K_s^*)$, or y and a vertex from $V(K_r^*) \cup V(K_s^*)$. It is not difficult to check that all digraphs in the family satisfy the condition in Theorem 3.1 and hence are Hamiltonian.

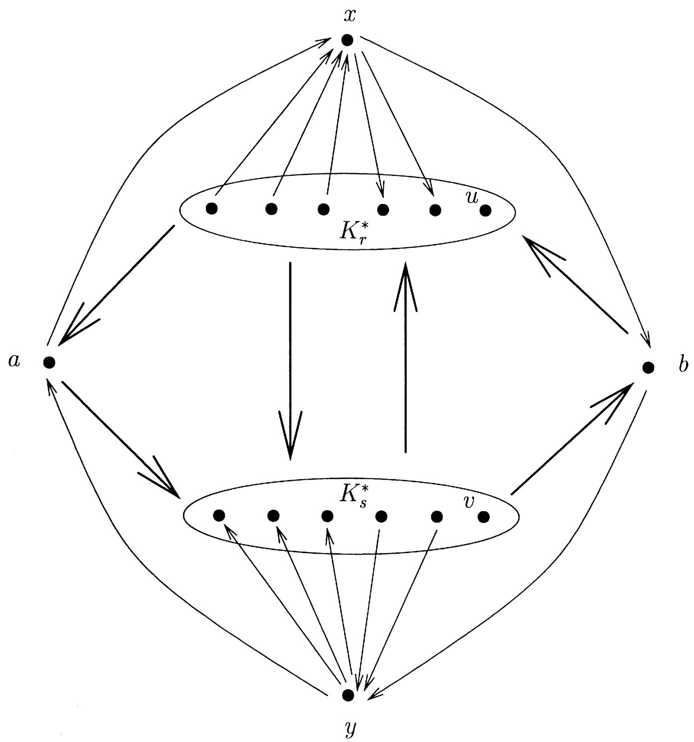

Fig. 1. The infinite family $D(r,s)$, $r,s \geqslant 6$. K_r^*, K_s^* denote complete digraphs on r,s vertices and large arcs indicate that all possible arcs in the shown direction are present. The vertex x has precisely three in-neighbours and two out-neighbours in $V(K_r^*)$ and is not adjacent to any other vertex of $V(K_r^*)$ (i.e. u is just one out of possibly many non-neighbours). Similarly y has precisely five neighbours in $V(K_s^*)$ as indicated.

The vertices x and y are not adjacent and have $d(x)+d(y)=14<2n-1$. Hence $D(r,s)$ satisfies neither Meyniels condition nor the condition in Theorem 1.5. The vertices x and u are non-adjacent and have a common out-neighbour and it is easy to check that $d^+(x)+d^-(u)=n-1$ and $d^-(x)+d^+(u)=n$. Thus $D(r,s)$ does not satisfy the condition of Theorem 1.6. Similarly, the vertices y and v are non-adjacent and have a common in-neighbour, so since $d^+(x)+d^+(u)+d^-(y)+d^-(v)=2n-2$, $D(r,s)$ also does not satisfy the condition in Theorem 1.7. Finally, it is easy to see that $D(r,s)$ does not satisfy the condition in Theorem 1.4 by considering the vertices x,y,u.

5. Support for the conjecture

In this section we prove two results which provide some support for Conjecture 1.8. We start by proving that one obviously necessary condition for the existence of a Hamilton cycle is satisfied by all the digraphs considered in Conjecture 1.8, namely that all such digraphs have a factor. The following lemma is easy to prove using for example Hall's Theorem.

Lemma 5.1 (Gutin and Yeo [7] and Yeo [11]). *Let D be a digraph with no factor. Then we can partition $V(D)$ into subsets Y, Z, R_1, R_2 such that $R_1 \Rightarrow Y$, $(R_1 \cup Y) \Rightarrow R_2$, $|Y| > |Z|$ and Y is a set of independent vertices.*

Theorem 5.2. *Let D be a strong digraph of order n. Suppose that $d(x)+d(y) \geqslant 2n-1$ for every pair of non-adjacent vertices $\{x, y\}$ with a common out-neighbour or a common in-neighbour. Then there is a factor in D.*

Proof. Suppose that there is no factor in D and let Y, Z, R_1, R_2 be defined as in Lemma 5.1. Note that as D is strong we have $|Y| > |Z| \geqslant 1$. Without loss of generality, we assume that $|R_1| \leqslant |R_2|$ (otherwise we reverse all arcs). Let $Y = \{y_1, y_2, \ldots, y_m\}$ where $m = |Y|$ and let $D' = D\langle R_1 \cup Y\rangle$. For all $i = 1, 2, \ldots, m$ let A_i contain the vertices from $R_1 \cup Y$ which have a path to y_i in D', but no path in D' to any of the vertices in $Y - y_i$. Let $A_\emptyset$ contain those vertices from $R_1 \cup Y$ which do not have a path in D' to any of the vertices in Y. Let A_∞ contain the vertices from $R_1 \cup Y$ which have a path to at least two of the vertices in Y. We note that $A_1, A_2, \ldots, A_m, A_\emptyset, A_\infty$ partition the vertices in $R_1 \cup Y$. Furthermore, note that $y_i \in A_i$ for all $i = 1, 2, \ldots, m$.

Observe that, by definition, $A_\infty \Rightarrow A_i \Rightarrow A_\emptyset$, $i = 1, 2, \ldots, m$ and $A_\infty \Rightarrow A_\emptyset$. Furthermore there are no arcs between the sets A_i and A_j when $i \neq j$. Let $i \neq j$ and let $a_i \in A_i$ and $a_j \in A_j$ be arbitrary. Note that we have

$$d(a_i) + d(a_j) \leqslant 2n - 2. \tag{14}$$

This is seen as follows:

$$\begin{aligned} d(a_i) + d(a_j) &\leqslant 2(|A_i| - 1) + |A_\infty| + |A_\emptyset| + 2|Z| + |R_2| \\ &\quad +2(|A_j| - 1) + |A_\infty| + |A_\emptyset| + 2|Z| + |R_2| \\ &= 2(|A_i| - 1) + 2(|A_j| - 1) + 2|A_\infty| + 2|A_\emptyset| \\ &\quad +2|Z| + 2(|Z| + 1) - 2 + 2|R_2| \\ &\leqslant 2|A_i - Y| + 2|A_j - Y| + 2|A_\infty| + 2|A_\emptyset| + 2|Z| + 2|Y| - 2 + 2|R_2| \\ &\leqslant 2n - 2. \end{aligned}$$

Suppose first that $A_\infty \neq \emptyset$ and let $a \in A_\infty$ be arbitrary. Let $i \neq j$ be defined such that a has a path to both y_i and y_j in D'. By (14) we note that for each vertex $w \in A_\emptyset \cup R_2$, w cannot be dominated by vertices from both A_i and A_j, and for each vertex $r \in A_\infty$, r cannot dominate vertices from both A_i and A_j. This implies that $|N(A_i) - Z| + |N(A_j) - Z| \leqslant |A_\emptyset| + |A_\infty| + |R_2|$. Without loss of generality, assume that $|N(A_i) - Z| \leqslant \frac{1}{2}(|A_\emptyset| + |A_\infty| + |R_2|)$. Let $Q = N^-(A_i) \cap A_\infty$, and observe that $Q \neq \emptyset$, since a can reach y_i in D'. By (14) and the definition of A_∞ there exists vertices

$q \in A_\infty$, $p \in A_\infty$ and $a_i \in A_i$ such that $q \rightarrow \{p, a_i\}$ but p has no arc to A_i. By (14) p can have arcs into at most one of the sets $A_1, A_2, \ldots, A_m$, so assume, without loss of generality, that p has arcs into the set A_k (if p has no arcs to any A_b, then just choose $k \neq i$). This gives us the following contradiction (using that $|R_2| \geqslant |R_1|$):

$$\begin{aligned} d(a_i) + d(p) &\leqslant 2(|A_i| - 1) + 2|Z| + \frac{|A_\emptyset| + |A_\infty| + |R_2|}{2} \\ &\quad +2(|A_\infty| - 1) + |A_k| + |A_\emptyset| + 2|Z| + |R_2| \\ &= 2|A_i - Y| + |A_k| + 4|Z| + 2|A_\infty| + 2|R_2| \\ &\quad -2 + |A_\emptyset| - \frac{|R_2| - |A_\emptyset| - |A_\infty|}{2} \\ &\leqslant 2|A_i - Y| + 2|A_k - Y| + 2 + 2|Z| + 2(|Z| + 1) - 2 \\ &\quad +2|A_\infty| + 2|A_\emptyset| + 2|R_2| - 2 \\ &\leqslant 2n - 2. \end{aligned}$$

Therefore $A_\infty = \emptyset$. Since D is strong and $|Z| < |Y|$ there must be a vertex in Z which dominates a vertex in two distinct sets A_i and A_j. However, this is a contradiction against (14). Thus in all cases we reach a contradiction to the assumption that D has no factor and the proof is complete. □

The following digraph D shows that the condition $d(x) + d(y) \geqslant 2n - 1$ in Theorem 5.2 cannot be weakened: $V(D)$ is the disjoint union of sets A, B, Y, Z all non-empty and $|Y| = |Z| + 1$. Each of A, B, Z induce complete digraphs, Y contains no arcs, $A \Rightarrow Y \cup B$, $Y \Rightarrow B$ and there are all possible arcs between Y and Z, between A and Z and between Z and B. The only pairs of non-adjacent vertices are $y, y' \in Y$ and it is easy to check that we have $d(y) + d(y') = 2n - 2$ for all choices of $y, y' \in Y$.

Now we show that if we strengthen the degree condition in Conjecture 1.8 somewhat, then we get a sufficient condition for Hamiltonicity.

Lemma 5.3. *Let D be a strong digraph with $n \geqslant 2$ vertices. Suppose that $d(x) + d(y) \geqslant \frac{5}{2}n - 4$ for every pair of non-adjacent vertices x, y with a common in-neighbour or a common out-neighbour. Then D is Hamiltonian.*

Proof. Let $C = x_1 x_2 \ldots x_m x_1$ be a longest cycle in D and suppose that $m < n$. It is easy to see from the analogous calculations in the proof of Theorem 3.1 that D contains a C-bypass. In fact, it also follows from these calculations that already the condition in Conjecture 1.8 are sufficient for this. Let $P = u_1 u_2 \ldots u_s$ ($s \geqslant 3$) be a C-bypass with minimum gap among the gaps of all C-bypasses. Assume without loss of generality, that P is minimal respect to the minimum gap. In the following, we use the notation from the proof of Theorem 3.1. By the same arguments as in that proof, we have $d_R(u_{s-1}) + d_R(x_{\gamma-1}) \leqslant 2(n - m - 1)$, $\gamma \geqslant 5$ and $d_{C''}(u_{s-1}) \leqslant |V(C'')| + 1$. It follows from the assumption of the theorem that

$$\begin{aligned}5n/2-4 &\leqslant d(u_{s-1})+d(x_{\gamma-1})\\ &= d_{C''}(u_{s-1})+d_R(u_{s-1})+d_R(x_{\gamma-1})+d_{C'}(x_{\gamma-1})+d_{C''}(x_{\gamma-1})\\ &\leqslant (|V(C'')|+1)+2(n-m-1)+2(|V(C')|-1)+d_{C''}(x_{\gamma-1})\\ &= 2n-|V(C'')|-3+d_{C''}(x_{\gamma-1}).\end{aligned}$$

It follows that

$$\begin{aligned}d_{C''}(x_{\gamma-1}) &\geqslant |V(C'')|+n/2-1\\ &\geqslant |V(C'')|+(|V(C'')|+4)/2-1\\ &= \tfrac{3}{2}|V(C'')|+1.\end{aligned}$$

Similarly, we can deduce that

$$d_{C''}(x_2) \geqslant \tfrac{3}{2}|V(C'')|+1.$$

Adding these two equations and using that $d^-_{C''}(x_{\gamma-1})+d^+_{C''}(x_2)\leqslant 2|C''|$, we get that $d^-_{C''}(x_2)+d^+_{C''}(x_{\gamma-1})\geqslant |V(C'')|+2$. However this implies, as we argued in the proof of Theorem 3.1 that $C[x_2,x_{\gamma-1}]$ can be inserted in C'' contradicting the maximality of C. □

Note that it is possible, by more involved arguments, to improve on the constant part of the condition, but our approach does not seem to allow a better constant factor on n.

6. Concluding remarks

As we pointed out in the introduction, it seems quite difficult to prove Conjecture 1.8. Hence it may be of interest to consider the following weakening of the conjecture, which still does not seem to follow easily from our results so far.

Conjecture 6.1. *Let D be a strong digraph. Suppose that $d(x)+d(y)\geqslant 2n-1$ for every pair of non-adjacent vertices $\{x,y\}$ with a common out-neighbour or a common in-neighbour. Then D has a Hamiltonian path.*

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